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2004 Phys.-Usp. 47 821

(http://iopscience.iop.org/1063-7869/47/8/A04)

METHODOLOGICAL NOTES PACS numbers: 03.50.De, 05.45.Ac, **41.20.** – **q**, 64.60.Ak, **84.32.** – **y**

Stability and chaos in randomly inhomogeneous two-dimensional media and *LC* circuits

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DOI: 10.1070/PU2004v047n08ABEH001678

Contents

Abstract. Electric field distribution in 2D self-dual media, especially in nearly percolating media and *LC* circuits (low-frequency filters), is viewed both from the physical (fluctuations and absorption) and mathematical (stability) standpoints. The finite energy absorption paradox occurring in such systems of nonabsorbing elements is discussed.

1. Introduction

In these notes we examine two physical phenomena in the electrodynamics of inhomogeneous media that have recently attracted a lot of attention. One is the anomalously large spatial fluctuations of local electric fields in randomly inhomogeneous two-dimensional media at the percolation threshold. The other is the anomalously large absorption of energy in such media. In both cases we are dealing with two-dimensional two-phase inhomogeneous media near and at the percolation threshold, i.e., at critical phase concentration p_c, in conditions where the real part of the local conductivity of the phases is small (ideally, equal to zero), while the imaginary parts have different signs. It can be said that the medium consists of inductance coils and capacitors, which are characterized by permeability and permittivity. Films consisting of metallic islets separated by dielectric regions constitute an example of such media. The metallic areas of the film possess inductances, while the dielectric areas (often simply air) possess capacitances. The first exact solution to the two-dimensional problem of the effective conductivity of a two-phase medium (film) at equal concentrations of the phases and their geometrically equivalent (on the average) arrangement was found in Ref. [1]. In this case, the restrictions on the phase conductivities were not imposed. In particular, one of the phases may constitute an ideal conductor, so that conductance has only a negative imaginary part (inductance), while the second phase may constitute an ideal capacitor (conductance has only a positive imaginary part). According to Ref. [1], the conductance of such a medium is on the whole real, i.e., a medium "consisting of imaginary resistors not leading to energy dissipation has a real-valued equivalent resistivity, i.e., absorbs energy". This paradox is resolved if we allow for the fact that "the source energy is expended on resonantly exciting local vibrations. Here, the presence of absorption, no matter how small, in the system will lead to true (finite) energy dissipation" [1]. For media of dimensions smaller than the correlation radius, huge fluctuations in the inhomogeneity of the local fields are observed (e.g., see Refs [2 – 4] and the review [5]). The above-mentioned paradox, the 'emergence' of a real-valued part on the resistivity of a medium consisting of elements with a purely imaginary part in their resistivity [1], is also present in the simplest circuit theory comprising the theory of a ladder-type filter (*LC* circuits) and continues to draw attention [6]. Diametrically opposite opinions concerning the existence of a real-valued part of resistivity in a ladder-type filter within a certain frequency range can be found in college physics courses, which results in quite different explanations of how such filters operate.

We begin, in Section 2, by examining the problem of finding the effective conductivity of a two-dimensional two-phase medium at the percolation threshold in the case of a small or zero dissipative part in the phase conductivity. In Section 3, we use the example of an implementation of such a medium to show that there is deterministic chaos. In Section 4, we examine the behavior of the conductivity of a hierarchic network implementation of such media. Section 5 is devoted

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Received 15 May 2003, revised 14 January 2004
Uspekhi Fizicheskikh Nauk **174** 887 – 894 (2004)
Translated by E Yankovsky; edited by A Radzig

to a discussion. Finally, in Section 6 we briefly examine the problem of the impedance of a ladder-type circuit.

2. Two-dimensional two-phase media with a dual phase arrangement

For two-phase media with a geometrically equivalent (on the average) arrangement of the phases, the following exact expression for the effective conductivity σ_e was derived in Ref. [1]:

$$\sigma_e = \sqrt{\sigma_1 \sigma_2}\,. \quad (1)$$

The phases with conductivities σ_1 and σ_2 in such media are dual, i.e., interchanging of the phase conductivities $\sigma_1 \leftrightarrow \sigma_2$ does not affect the effective conductivity σ_e. Examples of such dual media, which we call D-media for brevity, comprise two-dimensional randomly inhomogeneous media at the percolation threshold $p_c = 0.5$. We are speaking, of course, of the case where for a randomly inhomogeneous medium one can introduce the concept of effective conductivity, in particular, for media whose dimensions $L > \xi$, where ξ is the correlation length — that is, the length over which self-averaging occurs (in our case, the self-averaging of effective conductivity). By self-averaging we mean that in calculating (for each random implementation) a quantity characterizing the system on the whole, no additional averaging over implementations is required.

Expression (1) was derived on the basis of the following symmetry transformations first introduced in Ref. [1]:

$$\tilde{\mathbf{j}} = \Lambda \mathbf{n} \times \mathbf{E}\,, \quad \tilde{\mathbf{E}} = \Lambda^{-1} \mathbf{n} \times \mathbf{j}\,, \quad (2)$$

where $\mathbf{n}$ is a unit vector normal to the plane of the medium, $\mathbf{j}$ and $\mathbf{E}$ are the current density and electric field strength in the 'parent' medium, respectively, and $\tilde{\mathbf{j}}$ and $\tilde{\mathbf{E}}$ are the analogous quantities in the dual medium (Fig. 1). The constant Λ is found from the requirements that $\tilde{\sigma}_e = \sigma_e$ and $\Lambda^2 = \sigma_1 \sigma_2$, and of course the fields and currents in the 'parent' and dual media are related through Ohm's law: $\mathbf{j} = \sigma(\mathbf{r})\mathbf{E}$ and $\tilde{\mathbf{j}} = \tilde{\sigma}(\mathbf{r})\tilde{\mathbf{E}}$. From the volume-averaged transformations (2), namely

$$\langle \tilde{\mathbf{j}} \rangle = \Lambda \mathbf{n} \times \langle E \rangle\,, \quad \langle \tilde{\mathbf{E}} \rangle = \Lambda^{-1} \mathbf{n} \times \langle \mathbf{j} \rangle\,, \quad (3)$$

where $\langle \ldots \rangle$ stands for an average over a volume whose characteristic size is greater than the correlation length, it follows that $\sigma_e = \Lambda$, which with allowance for $\Lambda^2 = \sigma_1 \sigma_2$ yields (1).

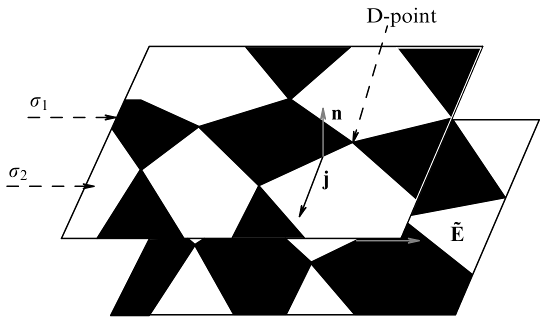

Figure 1. Two-phase dual medium. At the top is the 'parent' medium, and at the bottom is the dual medium, with **n** being a unit vector normal to the plane of the medium. The dashed arrow points to one of the D-points. The area with conductivity $\sigma_1(\sigma_2)$ in the 'parent' medium corresponds to the area with $\sigma_2(\sigma_1)$ in the dual medium.

When both phases have purely imaginary conductivities of different signs, viz.

$$\sigma_1 = -\mathrm{i}y\,, \quad \sigma_2 = \mathrm{i}x\,, \quad x > 0\,, \quad y > 0\,, \quad (4)$$

where $x = \omega c$, and $y = 1/\omega l$, with c and l being the specific capacitance and inductance of the material measured, respectively, in farads and henries per cubic meter, equation (1) yields

$$\sigma_e = \sqrt{xy} = \sqrt{\frac{c}{l}}\,. \quad (5)$$

According to the last relationship, a medium consisting of nondissipative elements ($\mathrm{Re}\,\sigma_1 = \mathrm{Re}\,\sigma_2 = 0$) is dissipative with $\mathrm{Re}\,\sigma_e > 0$, which constitutes the paradox.

3. Hierarchic implementation of a D-medium

Let us examine an implementation of a D-medium obtained through what is known as the mixing procedure [7–10]. In the first stage (Fig. 2a), the medium is 'built up' from strips of thicknesses d_1 and d_2 and conductivities σ_1 and σ_2, with, of course, $p/d_1 = (1-p)/d_2$, where p is the concentration of the phase with the conductivity σ_1. Turning the strip thicknesses to zero, i.e., homogenizing the medium, we get a 'single crystal' (Fig. 2b) with principal components of the conductivity tensor

$$\sigma_{\parallel}^{(1)} = p\sigma_1 + (1-p)\sigma_2\,, \quad \sigma_{\perp}^{(1)} = \frac{\sigma_1 \sigma_2}{(1-p)\sigma_1 + p\sigma_2}\,. \quad (6)$$

Then, cutting this 'single crystal' in longitudinal and transverse directions into strips of thicknesses d_1 and d_2, respectively, we fabricate a new 'single crystal' (Fig. 2b), whose principal components of the conductivity tensor are $\sigma_{\parallel}^{(2)}$ and $\sigma_{\perp}^{(2)}$. Doing this procedure n times, we notice at each stage that

$$\sigma_{\parallel}^{(n+1)} = p\sigma_{\parallel}^{(n)} + (1-p)\sigma_{\perp}^{(n)}\,, \quad \sigma_{\perp}^{(n+1)} = \frac{\sigma_{\parallel}^{(n)} \sigma_{\perp}^{(n)}}{p\sigma_{\parallel}^{(n)} + (1-p)\sigma_{\perp}^{(n)}}\,. \quad (7)$$

Since the iterative procedure (7) has an invariant

$$I = \sigma_{\parallel}^{(n+1)} \sigma_{\perp}^{(n+1)} = \sigma_{\parallel}^{(n)} \sigma_{\perp}^{(n)} = \sigma_1 \sigma_2 \frac{p\sigma_1 + (1-p)\sigma_2}{(1-p)\sigma_1 + p\sigma_2}\,, \quad (8)$$

the first equation in formulas (7) can be written down as

$$\sigma_{\parallel}^{(n+1)} = p\sigma_{\parallel}^{(n)} + (1-p)\,\frac{I}{\sigma_{\parallel}^{(n)}}\,. \quad (9)$$

In the 'ordinary' case with $\mathrm{Im}\,\sigma_1 = \mathrm{Im}\,\sigma_2 = 0$, the mixing procedure rapidly converges yielding the isotropic medium with

$$\sigma_e = \sqrt{\sigma_1 \sigma_2} \sqrt{\frac{p\sigma_1 + (1-p)\sigma_2}{(1-p)\sigma_1 + p\sigma_2}}\,. \quad (10)$$

For phase concentration $p = p_c = 1/2$, expression (10) is reduced to Eqn (1), and the iteration procedure amounts to

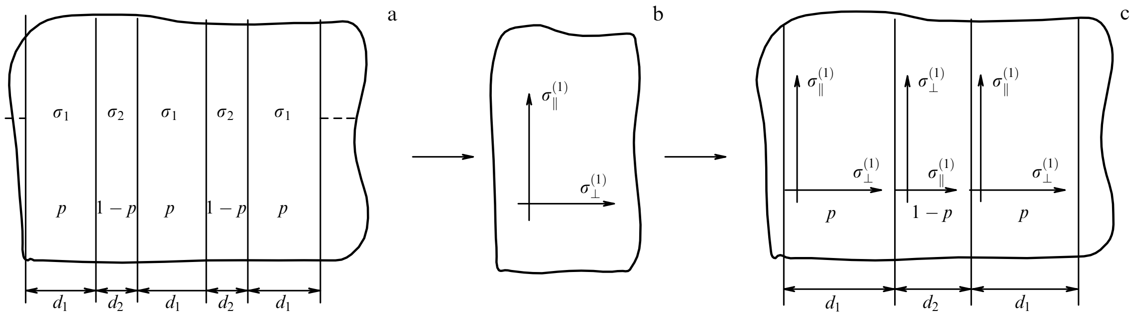

Figure 2. Procedure for the consecutive building (mixing) of a two-phase hierarchic medium with arbitrary phase concentration. In the first stage (a) the medium consists of strips of the first (σ_1) and second (σ_2) phases with concentrations p and $1-p$; after homogenization ($d_{1,2} \to 0$) the medium becomes a 'single crystal' (b) with principal components $\sigma_{\parallel}^{(1)}$ and $\sigma_{\perp}^{(1)}$ of the conductivity tensor. The next stage (c) consists in homogenization of the medium composed of strips cut from the 'single crystal' fabricated in the previous stage.

the method for calculating the square root by Newton's method, which is a common in applied mathematics (Fig. 3a). The inset to Fig. 3a shows how $\sigma_{\parallel}^{(n)}$ changes in the course of the iteration procedure. In the case of conductivities with a zero real part and imaginary parts of the same sign (e.g., the medium consists of two media with different dielectric constants), everything remains the same, to within notation. But when the imaginary parts have opposite signs, i.e., for $\mathrm{Re}\,\sigma_1 = \mathrm{Re}\,\sigma_2 = 0$ and $\mathrm{Im}\,\sigma_1\,\mathrm{Im}\,\sigma_2 < 0$, the situation

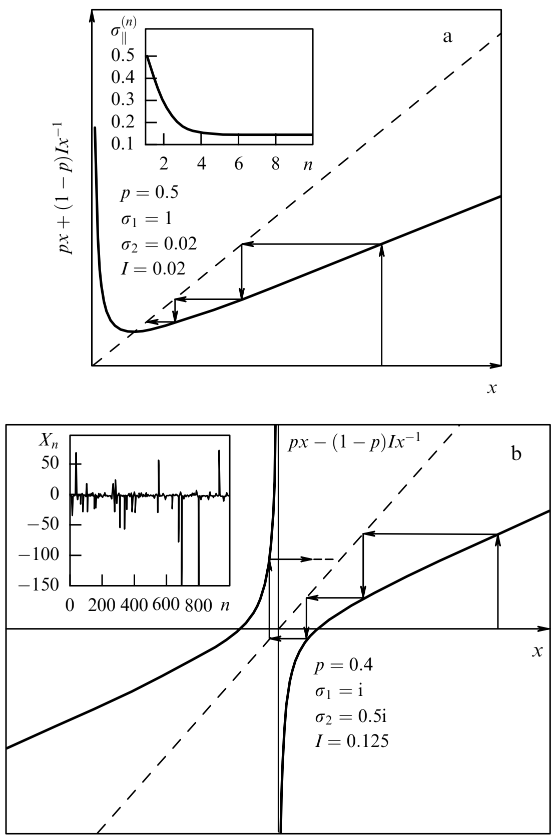

Figure 3. (a) One-dimensional mapping $x \to px + (1-p)\,I/x$ ($I = 0.02$, $\sigma_1 = 1$ and $\sigma_2 = 0.02$) of process (9) with concentration $p = 1/2$ in the presence of a fixed point. Clearly, there is rapid convergence to $\sqrt{I} = \sqrt{0.02} = 0.141$. (b) One-dimensional mapping $x \to px - (1-p)\,I/x$ ($I = 0.125$) of process (13) with concentration $p = 0.4$ and $\sigma_1 = \mathrm{i}$ and $\sigma_2 = -0.5\mathrm{i}$; the inset shows the iteration process for these parameters.

changes dramatically (Fig. 3b). Allowing for the fact that $\mathrm{Re}\,\sigma_{\perp}^{(n)} = \mathrm{Re}\,\sigma_{\parallel}^{(n)} = 0$ and introducing the notation for $\sigma_{\perp}^{(n)}$ and $\sigma_{\parallel}^{(n)}$ similar to formulas (4):

$$\sigma_{\perp}^{(n)} = -\mathrm{i}\,Y_n\,, \quad \sigma_{\parallel}^{(n)} = \mathrm{i}X_n\,, \quad \mathrm{Im}\,X_n = \mathrm{Im}\,Y_n = 0\,, \tag{11}$$

we arrive at an expression for the invariant I:

$$I = xy\,\frac{py - (1-p)\,x}{(1-p)\,y - px}\,, \tag{12}$$

and instead of the iteration process (9) we have the iteration process

$$X_{n+1} = pX_n - (1-p)\,\frac{I}{X_n}\,, \tag{13}$$

whose fixed point X^* is given by

$$X^* = \pm\sqrt{-I}\,. \tag{14}$$

According to definition (11), $\mathrm{Im}\,X_n = 0$, with the result that the fixed point X^* exists only if $I < 0$, which is possible if any of the two systems of inequalities

$$\begin{cases} py > (1-p)\,x \\ (1-p)\,y < px \end{cases}, \quad \begin{cases} py < (1-p)\,x \\ (1-p)\,y > px \end{cases} \tag{15}$$

is satisfied (Fig. 4). The hatched regions in Fig. 4 correspond to an empty set of solutions to systems (15), i.e., there is no fixed point.

Thus, on the one hand, geometrically the medium that exists at $p = 1/2$ is a D-medium but, on the other hand, as Fig. 4 shows, there is no fixed point. If we write down the mapping (9) in the form $N(z) = (z - I/z)/2$, $\mathrm{Im}\,I = 0$, $\mathrm{Re}\,I > 0$, it can be shown (see Ref. [11]) that the imaginary axis coincides with its Julia set J_N which separates the basins of attraction of stable fixed points $\pm\sqrt{I}$. The mapping $N(z)$ is conjugate to the mapping $R(u) = u^2$ obtained through the substitution $u = (z + \sqrt{I})/(z - \sqrt{I})$. Here, the imaginary axis (the Julia set) goes over into a unit circumference whereon the dynamics is given by the mapping $r(\theta) = 2\theta\,\mathrm{mod}\,2$. The latter, as is well known (see Refs [11, 12]), generates chaotic dynamics.

Hence, as is evident from the foregoing account, when the parameters of the medium land in the hatched area, the

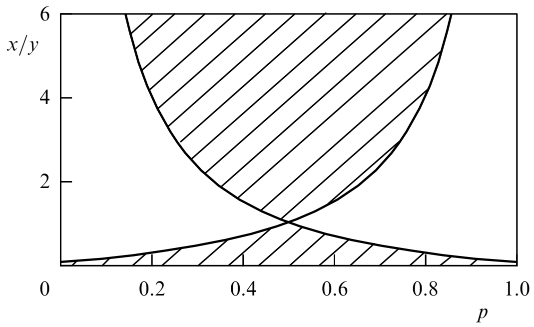

Figure 4. Plots of instability regions (hatched areas) of the mixing procedure (13) versus the first phase concentration p and the conductivity ratio x/y of the phases.

iteration process does not converge, and at $p = 0.5$ there occurs strongly deterministic chaos. This means, among other things, that self-averaging is lacking in the medium, i.e., the correlation radius tends to infinity.

4. Hierarchic D-network

Let us examine one more implementation of a D-medium, which is termed a hierarchic D-network (Fig. 5). After introducing notation for the initial (zero) stage of building up such a network, namely

$$R_0(r_x) = \frac{r_x r_1 + 2 r_1 r_2 + r_x r_2}{r_1 + 2 r_x + r_2}\,, \tag{16}$$

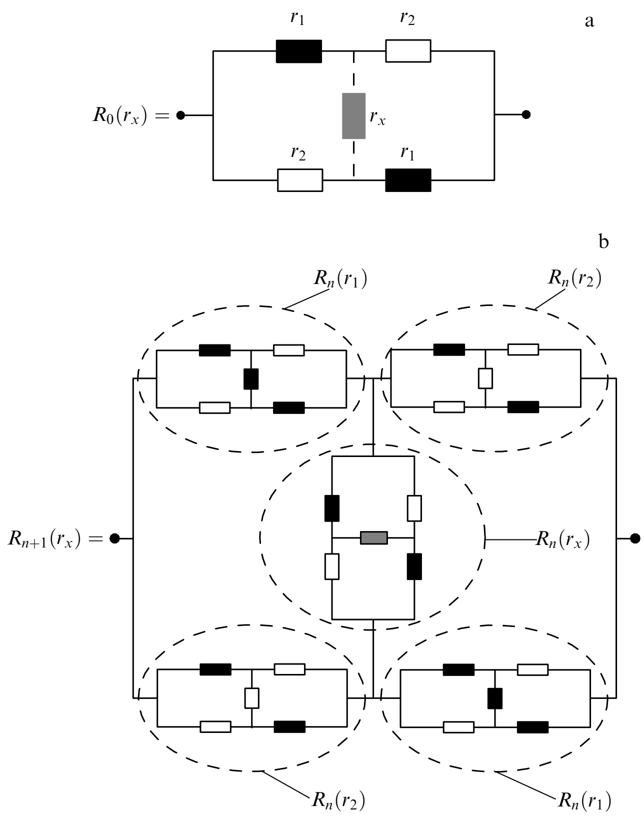

Figure 5. Hierarchic D-network. The initial (a) and nth (b) stages of building up such a network.

we can examine several different hierarchic networks $R_n(r_0)$, $R_n(r_1)$, and $R_n(r_2)$, which differ only in one resistor r_x at the 'center' of the network. As Fig. 5 suggests, the iteration processes for these three networks can be written in the following way:

$$R_{n+1}(r_0) = \frac{R_n(r_0)\,R_n(r_1) + 2R_n(r_1)\,R_n(r_2) + R_n(r_0)\,R_n(r_2)}{R_n(r_1) + 2R_n(r_0) + R_n(r_2)} \tag{17}$$

for $R_{n+1}(r_0)$, and similarly for $R_n(r_1)$ and $R_n(r_2)$, with r_1 and r_2 substituted for r_0 in formula (17), respectively. The network $R_n(r_0)$ at $r_x = r_0 = \sqrt{r_1 r_2}$ is a D-network for any number n, and its resistance is exactly $r_0 = \sqrt{r_1 r_2}$. At $\operatorname{Im} r_1 = \operatorname{Im} r_2 = 0$, one finds

$$\lim_{n\to\infty} R_n(r_1) = \lim_{n\to\infty} R_n(r_2) = \sqrt{r_1 r_2}\,, \quad \operatorname{Im} r_1 = \operatorname{Im} r_2 = 0\,, \tag{18}$$

as would be expected, since to within a single resistor (against the background of $n \to \infty$) the networks $R_n(r_1)$ and $R_n(r_2)$ are D-networks as well.

When $\operatorname{Re} r_1 = \operatorname{Re} r_2 = 0$ and $\operatorname{Im} r_1 \operatorname{Im} r_2 < 0$, the networks $R_n(r_1)$ and $R_n(r_2)$ also differ from a D-network by a single resistor, but here the transformation

$$R_{n+1}(r_1) = R_n(r_1)\,\frac{R_n(r_1) + 3R_n(r_2)}{3R_n(r_1) + R_n(r_2)}\,,$$

$$R_{n+1}(r_2) = R_n(r_2)\,\frac{3R_n(r_1) + R_n(r_2)}{R_n(r_1) + 3R_n(r_2)} \tag{19}$$

has no fixed stable point, and the behavior of both $R_n(r_1)$ and $R_n(r_2)$ is chaotic. If for $\operatorname{Im} r_1 \operatorname{Im} r_2 < 0$ the resistance r_x is set equal to real-valued resistance, $r_x = r_0 = \sqrt{r_1 r_2}$, i.e., a D-resistance, such a replacement of a single resistance in an arbitrarily large network will change the behavior of the network dramatically, since now a stable fixed point exists:

$$\lim_{n\to\infty} R_n(r_0) = r_0 = \sqrt{r_1 r_2}\,, \quad \operatorname{Im} r_1 \operatorname{Im} r_2 < 0\,. \tag{20}$$

Notice that if for r_x we take a real-valued resistance that is not equal to the D-resistance, i.e., $r_x \neq r_0 = \sqrt{r_1 r_2}$, the sequence $R_n(r_x, \operatorname{Im} r_x = 0)$ again begins to oscillate chaotically, but around the 'correct' value $r_0 = \sqrt{r_1 r_2}$, with the average value $\langle R_n(r_x) \rangle$ tending to r_0 as n increases:

$$\bigl\langle R_n(r_x) \bigr\rangle = \lim_{N\to\infty} \frac{1}{1+N} \sum_{n=0}^{N} R_n(r_x) \to r_0 = \sqrt{r_1 r_2}\,. \tag{21}$$

5. Discussion

Arkhincheev [13] proposed one further approach to the problem of stability in D-media. From formulas (3) and the definition of effective conductivity as the quantity that couples the volume averages of the fields and currents, it follows that $\tilde{\sigma}_e = \Lambda^2/\sigma_e$ [1]. Arkhincheev [13] interpreted this relationship as the transformation $f(Z) = \Lambda^2/Z$ in the complex domain whose fixed (stable or unstable) point $Z^* = \pm\Lambda$ is the effective value. According to Banach's contraction-mapping theorem (see, for example, Section 5.3 in Ref. [14]), the inequality $|f'(Z^*)| < 1$ establishes the criterion of stability of the fixed point Z^*. In our case

($\mathrm{Im}\,\sigma_1\,\mathrm{Im}\,\sigma_2 < 0$) we have $|f'(Z^*)| = 1$, i.e., Z^* is not a stable fixed point. Indeed, for any Z as close to Z^* as desired we have $f(f(Z)) = Z$ — that is, the mapping does not converge to Z^*, and so we are concerned with a focus rather than a stable center. In Ref. [13], the transformation $Z \to f(Z) = \Lambda^2/Z$ is first written in the finite-difference form

$$Z_{n+1} - Z_n = f(Z_n) - Z_n = F(Z_n)\,, \qquad (22)$$

after which it is written in the form of differential equations [for the real and imaginary parts of expression (22)] with respect to the independent variable n. If such a passage were meaningful, then, according to the stability theory of differential equations (e.g., see Section 1.2 in Ref. [14]), the stability criterion $\mathrm{Re}(F'(Z^*)) < 0$ would be met, since in the case at hand $\mathrm{Re}(F'(Z^*)) = -2$. However, the passage to the continuous case is not meaningful, which follows if only from the equality $Z_{n+2} = Z_n$.

For network implementations of D-media (here we are dealing with ordinary networks and not with hierarchic networks, which were examined in Section 4), the absence of a definite value of σ_e on a finite network for $\mathrm{Im}\,\sigma_1\,\mathrm{Im}\,\sigma_2 < 0$ was first emphasized by Helsing and Grimvall [15]. They found that at concentrations higher than p_c there is a continuous path from one contact to another (from '$-\infty$' to '∞') from both 'left to right' and 'bottom to top' along couplings that are inductance coils. However, the effective conductivity of a medium may be greater or smaller than zero, depending on the conductivities of the network elements, i.e., is either a capacitance or an inductance. In network media at concentrations not equal to the threshold one, the reciprocal relation $\sigma_e\tilde{\sigma}_e = \Lambda^2$ [1] means that if the effective conductivity of the 'parent' medium is of the capacitance type ($\mathrm{Im}\,\sigma_e > 0$), then that of the reciprocal medium is of the inductance type ($\mathrm{Im}\,\sigma_e < 0$). At concentrations tending to the threshold value, nothing changes and, in the final analysis, the effective conductivities of the 'parent' and dual media remain with different signs. On the other hand, in network media at the percolation threshold, if they are D-media, there is always at least one element (coupling) with a conductivity equal neither to the conductivity of the first phase nor the conductivity of the second phase (Fig. 6). The conductivity of this D-coupling is $\sqrt{\sigma_1\sigma_2}$ (see the details in Refs [15, 16]). In the continuum two-phase case of D-media, these couplings degenerate into points; the arrow in Fig. 1 points to just such a point. If there is no such a D-coupling, then for $\mathrm{Re}\sigma_{1,2} = 0$ and $\mathrm{Im}\,\sigma_1\,\mathrm{Im}\,\sigma_2 < 0$ the network is not a dual one, and, for one thing, the conductances from 'left to right' and from 'bottom to top' will be of opposite signs: along one direction, the medium will be a 'capacitor', and along another direction, an 'inductor'. But if there is a D-coupling, then, strictly speaking, the medium is not a D-medium consisting of two phases. The larger the medium, the smaller the effect of this coupling. This proves to be true for $\mathrm{Im}\,\sigma_1\,\mathrm{Im}\,\sigma_2 < 0$ only if $\mathrm{Re}\,\sigma_{1,2} \neq 0$.

Now let us see how absorption emerges in a D-medium as $\xi \to \infty$ at $\mathrm{Re}\,\sigma_{1,2} = 0$, i.e., how effective conductivity takes on a real value. The answer is simple: as $\xi \to \infty$ the conductivity fluctuates and its real part is finite. Clearly, the addition of a real term (as small as desired) to σ_1 or σ_2 makes the iteration process (13) discussed in Section 3 stable. Now, the process converges to the real value

$$\sqrt{\sigma_1\sigma_2} = \sqrt{(\mathrm{Re}\,\sigma_1 + \mathrm{i}\mathrm{Im}\,\sigma_1)(\mathrm{Re}\,\sigma_2 + \mathrm{i}\mathrm{Im}\,\sigma_2)} \approx \sqrt{-\mathrm{Im}\,\sigma_1\,\mathrm{Im}\,\sigma_2}$$

(in which we can disregard small 'seeding' real-valued terms). Figure 7 shows how small fluctuations lead out the phase point (in the $\{\mathrm{Re}\,\sigma - \mathrm{Im}\,\sigma\}$ space) from the $\mathrm{Im}\,\sigma$-axis, where it performed chaotic jumps (Fig. 3b), onto a path that converges to the real axis. Thus, we have two limiting processes $\xi \to \infty$ and $\mathrm{Re}\,\sigma_{1,2} \to 0$ that occur simultaneously, and these processes cannot be interchanged. For a finite-size medium, the number of elements (couplings and size of the medium) will compete with the real part of the conductivity. If the number of elements 'loses' this competition, there is no self-averaging in the medium and the given random fractal implementation occurs (the reader will recall that media at the percolation threshold have a fractal structure) [17, 18]. Naturally, in such a medium, which on

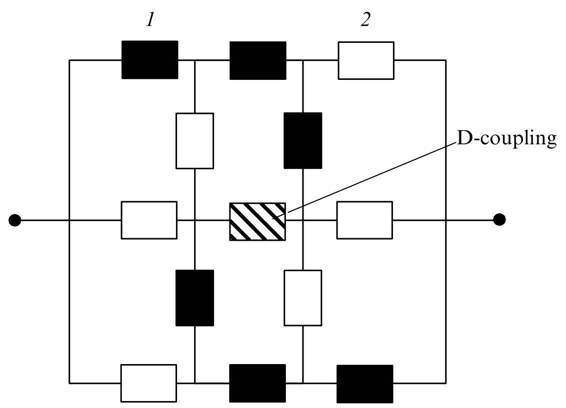

Figure 6. One possible implementation of a finite-size D-network [15]. Black rectangles stand for resistors z_1, and white rectangles stand for resistors z_2. At the center of the network there is a D-coupling, which in order to meet the network variant of the transformations (2) and (3) and the condition $\tilde{\sigma}_e = \sigma_e$ must have a resistance equal to $\sqrt{z_1 z_2}$. The D-coupling (resistor) is a network analog of D-points in the continuous case (see Fig. 1).

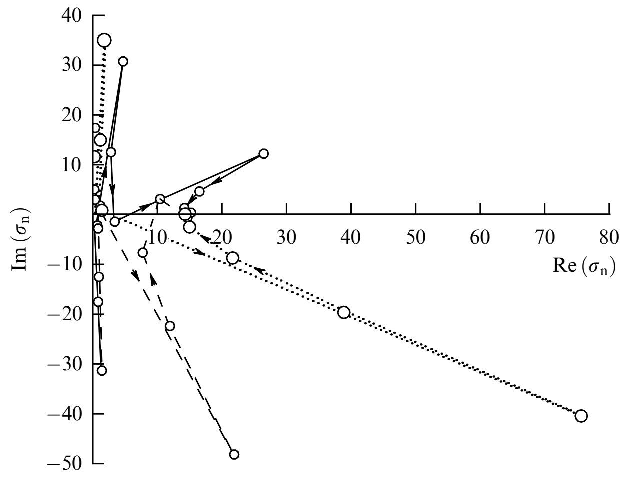

Figure 7. 'Phase' paths of the iteration process (9) in the $\{\mathrm{Im}\,\sigma - \mathrm{Re}\,\sigma\}$ space for small real 'seeding' parts in the phase conductivities and a concentration $p = 0.5$. The broken dashed line corresponds to $\sigma_1 = 0.038 - 5\mathrm{i}$ and $\sigma_2 = 0.1 + 40\mathrm{i}$; the solid line corresponds to $\sigma_1 = 0.1 - 10\mathrm{i}$ and $\sigma_2 = 0.1 + 20\mathrm{i}$, and the dotted line corresponds to $\sigma_1 = 0.056 - 6.667\mathrm{i}$ and $\sigma_2 = 0.1 + 30\mathrm{i}$. The phase paths rapidly converge to the real axis, precisely, to $\sqrt{\sigma_1\sigma_2} = 2\sqrt{(\mathrm{Re}\,\sigma_1 + \mathrm{i}\mathrm{Im}\,\sigma_1)(\mathrm{Re}\,\sigma_2 + \mathrm{i}\mathrm{Im}\,\sigma_2)} \approx \sqrt{-\mathrm{Im}\,\sigma_1\,\mathrm{Im}\,\sigma_2} \approx 14.14$, and the small 'seeding' parts in the initial conductivities can be ignored in the process.

the average is inhomogeneous, there are large spatial fluctuations, including fluctuations of Joule heat release, absorption, higher moments of the current distribution, and so forth [2–5].

It goes without saying that more refined methods are needed in order to do quantitative calculations of the distribution of local fields and their moments. Sarychev and Shalaev [5] found that the problem of the potential's distribution coincides with that of Anderson localization on a network with purely imaginary conductivities of the couplings. A small real term added to the imaginary part of the conductivity of the metallic phase can be taken into account perturbatively. Such an approach has been used, in particular, to calculate the correlation length which proved to be proportional to $1/\sqrt{\mathrm{Re}\,\sigma_1}$ (for the two-dimensional case). Thus, in the case of purely imaginary phases, the correlation length diverges: $\xi(\mathrm{Re}\,\sigma_1 \to 0) \to \infty$, a result that also follows from the above reasoning.

6. Ladder-type filter (*LC* circuit)

Now, let us turn to the problem of the impedance of a ladder-type *LC* circuit, a well-known problem from college physics courses (Fig. 8). What is amazing is that even the standard physics courses, such as Feynman et al. [19] and Sivukhin [20], give different solutions to this problem. And this is even more amazing if we account for the fact that an *LC* circuit is the simplest possible filter used in innumerable real devices.

The impedance Z of an infinite ladder-type circuit can be found by building such a circuit one element after another and writing the expression for Z_{n+1} as

$$Z_{n+1} = f(Z_n)\,, \quad f(Z_n) = z_1 + \frac{Z_n z_2}{Z_n + z_2}\,, \quad n = 1, 2, \ldots, \infty\,, \tag{23}$$

where z_1 and z_2 are complex-valued resistances, or conductances, of the circuit elements.

For all values of z_1 and z_2 there is always a fixed point Z^* determined by the equation $Z^* = f(Z^*)$:

$$Z^* = \frac{z_1}{2} \pm \sqrt{\frac{z_1^2}{4} + z_1 z_2}\,. \tag{24}$$

If the fixed point Z^* is stable, an infinite ladder-type chain has an impedance Z that is found from a passage to the limit $\lim_{n\to\infty} Z_n = Z^*$. In the opposite case (an unstable fixed point), the limit ($\lim_{n\to\infty} Z_n$) does not exist (Fig. 9; see a similar figure in Ref. [6]) and it is meaningless to speak of the impedance of an infinite chain.

Analysis of formulas (23) shows that in the ideal case of purely imaginary impedances (the chain consists of inductance coils and capacitors with zero ohmic resistances) for certain values of z_1 and z_2 there is no fixed stable point Z^*.

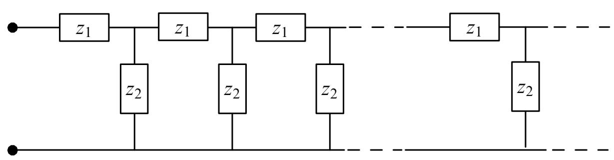

Figure 8. Infinite ladder-type *LC* circuit (filter) with $z_1 = \mathrm{i}\omega L$ and $z_2 = 1/\mathrm{i}\omega C$.

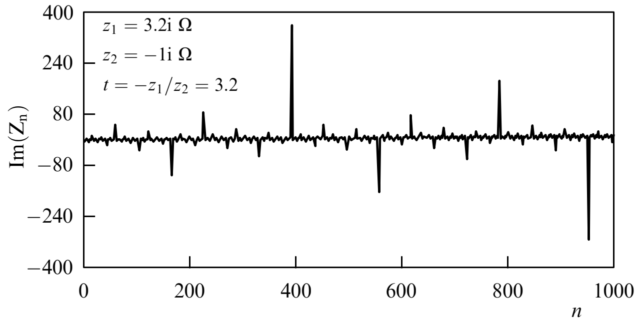

Figure 9. Chaotic dependence of the impedance of a ladder-type circuit on the number n of elements. The parameter t is selected equal to $-z_2/z_2 = 3.2$ and, according to the criterion (28), is within the instability range.

Indeed, the fixed point Z^* of the iteration process (23) is stable [12] if

$$\left|\frac{\mathrm{d}f(Z_n)}{\mathrm{d}Z_n}\right|_{Z_n = Z^*} < 1\,. \tag{25}$$

In our case, one obtains

$$\left|\frac{\mathrm{d}f(Z_n)}{\mathrm{d}Z_n}\right|_{Z_n = Z^*} = \left|\frac{1}{(1 + Z^*/z_2)^2}\right| = \frac{4}{\left|\left(2 + z_1/z_2 \pm \sqrt{(z_1/z_2)^2 + 4z_1/z_2}\,\right)^2\right|}\,. \tag{26}$$

Introducing the notation $t = -z_1/z_2$, we can rewrite condition (25) as follows:

$$F(t) = \frac{4}{\left|\left(2 - t \pm \sqrt{t^2 - 4t}\right)^2\right|} < 1\,. \tag{27}$$

For t real, the behavior of the function $F(t)$ in the range

$$0 \leqslant t \leqslant 4 \tag{28}$$

is somewhat unexpected (this can easily be checked) — it is independent of t and is equal to exactly 1. Thus, the inequality (27) is not met and the fixed point is not stable. Note that the addition of a real term (as small as desired) to z_1 or z_2 always makes the fixed point stable, which can be explained by the fact that a system with $F(t) = 1$ is on the verge of stability, and only an arbitrarily small shift is needed for the iteration sequence (23) to acquire a fixed stable point. The stable point exists for $t < 0$ and is given by

$$Z^* = \frac{1}{2}\left(z_1 + \sqrt{z_1^2 + 4z_1 z_2}\right), \quad t < 0\,, \tag{29}$$

while for $t > 4$ the stable point resides at

$$Z^* = \frac{1}{2}\left(z_1 - \sqrt{z_1^2 + 4z_1 z_2}\right), \quad t > 4\,. \tag{30}$$

Notice the different signs in front of the square root[1] in formulas (29) and (30).

[1] Van Enk [6] failed to notice that in calculating the fixed point for $t > 4$, the sign in front of the square root must be minus.

Let us examine in greater detail the case of pure imaginaries z_1 and z_2 with opposite signs. When $z_1 = \mathrm{i}\omega L$ and $z_2 = 1/\mathrm{i}\omega C$, expressions (27) and (29) suggest that for

$$\omega > \omega_0 = \frac{2}{\sqrt{LC}} \tag{31}$$

the fixed point is stable and the impedance of the infinite circuit exists and, as expected, is purely imaginary:

$$Z^* = \mathrm{i}\left(\frac{\omega L}{2} + \sqrt{\frac{\omega^2 L^2}{4} - \frac{L}{C}}\right). \tag{32}$$

For the frequency range

$$\omega < \omega_0 = \frac{2}{\sqrt{LC}}\,, \tag{33}$$

there is no fixed stable point and, hence, the concept of the impedance of an infinite circuit becomes meaningless. Therefore, the statement made by Feynman et al. in the lecture course [19] that by looking at an infinite circuit from the terminal a' one would see the characteristic impedance $Z_0 = \sqrt{L/C - \omega^2 L^2/4}$ is erroneous. To continue, the same authors claim that the impedance at low frequencies is pure resistance and, therefore, energy is absorbed. Thus, according to Feynman et al. [19], the transmission of a filter at low frequencies is related to the absorption of energy at these frequencies caused by dissipation ($\operatorname{Re} Z_0 > 0$). Later, however, the authors make a quite valid statement that when the source is coupled to the circuit, it must first supply energy to the first inductor and capacitor pair, then to the second pair, third, and so on. In circuits of this type energy constantly (and at a permanent rate) is 'sucked' from the generator and continuously flows into the circuit. The energy is stored in the inductance coils and capacitors.

We can say that the filter absorbs energy but that there is no dissipation, i.e., we must clearly distinguish between the absorption of energy by a medium consisting of LC elements with dissipation and without dissipation. The point is that in Ref. [19] the characteristic impedance Z_0 was found by solving a quadratic equation under the (incorrect) assumption that the impedance of a network consisting of n sections converges (for $\omega < \omega_0$). Van Enk [6] examined the behavior of Z_n in the limit of $n \to \infty$ as a function of the values of z_1 and z_2. As expected, at $z_1 = \mathrm{i}\omega L$ and $z_2 = 1/\mathrm{i}\omega C$ in the $\omega < \omega_0$ range the impedance Z_n converges. The range $\omega < \omega_0$ conforms to the filter transmission band. In Ref. [6], the filter transmission in this range is related (as in Ref. [19]) to the presence of dissipation in the system. In answering his own question: "How can equation (29) give [2] the right answer in practice when we just argued that this equation is incorrect?", van Enk says that "a *realistic* inductor has an internal resistance $r \neq 0$". In this case, the problem of the convergence of Z_n as $n \to \infty$ is, of course, resolved. If z_1 and/or z_2 comprises an arbitrarily small real part, the characteristic impedance $Z_0 = \sqrt{L/C - \omega^2 L^2/4}$ exists. However, in realistic filters with a finite number of elements (often this number is very small), a small real part in z_1 and/or z_2 can, obviously, change nothing. Hence, the 'existence' of $Z_0 = \sqrt{L/C - \omega^2 L^2/4}$ cannot be used as an explanation for the transmission of a filter.

[2] Here we have changed the number of the equation in accordance with our system of numbering equations.

Actually, and this is a well-known fact (e.g., see Ref. [21]), for a finite filter consisting of n sections with purely imaginary elements there are two solutions that relate the input voltage $U(t) = U_0 \cos\omega t$ and the output voltage $U_n(t)$. The first solution, valid for the transmission band, has the form

$$U_n(t) = U_0\,\frac{\cos\beta/2}{\cos(n+1/2)\beta}\cos\omega t\,,$$
$$\cos\beta = 1 - LC\omega^2\,,\quad \omega < \omega_0 = \frac{2}{\sqrt{LC}}\,. \tag{34}$$

At frequencies $\omega \ll \omega_0$, we have $U_n(t) \approx U_0 \cos\omega t$, i.e., a filter consisting of purely reactive elements with a finite number of constituents (this number can be very small) transmits the signal without distortions. At other frequencies belonging to the range $\omega < \omega_0$, the transmission is nonuniform, but still the system is not 'cut off'.

The second solution in the range $\omega > \omega_0$ takes the form

$$U_n(t) = U_0(-1)^n \times \frac{\exp(\xi/2) - \exp(-\xi/2)}{\exp\left[(n+1/2)\xi\right] - \exp\left[-(n+1/2)\xi\right]}\cos(\omega t)\,, \tag{35}$$

where ξ can be found by solving the equation $\cosh\xi = |1 - LC\omega^2/2|$.

Relation (35) clearly shows that the amplitude of the output signal decreases exponentially as the number n of sections in the circuit grows, and when this number is sufficiently large, we can write

$$U_n(t) = U_0(-1)^n\left[\exp\left(\frac{\xi}{2}\right) - \exp\left(-\frac{\xi}{2}\right)\right] \times \exp\left[-\left(n+\frac{1}{2}\right)\xi\right]\cos(\omega t)\,, \tag{36}$$

that is, the signal is blocked and the filter does not transmit a signal with frequencies higher than the critical ones.

A qualitative description of the operation of a filter amounts to the following. Within the transmission band there are resonances, and in a finite circuit the transmitted frequencies are close to the resonances, but in an infinite circuit the resonances merge and all transmitted frequencies coincide with resonance frequencies.

A A Snarskiĭ and M I Zhenirovskiĭ are grateful to P M Tomchuk and S P Luk'yanets for the numerous considerations of the topics covered in this article, and to A N Selin for discussing the mathematical aspects of the theory of stability of discrete systems. The work was partially supported by Deutscher Akadémisch Austausch Dienst e.V. (grant 322, A/02/16226-2002).

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